

Some Restrictions on the Symmetry Groups of Axially Symmetric Spacetimes

Alan Barnes

Computer Science Group, School of Engineering and Applied Science,
Aston University, Aston Triangle, Birmingham B4 7ET, UK †

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Abstract. Lie transformation groups containing a one-dimensional subgroup acting cyclically on a manifold are considered. The structure of the group is found to be considerably restricted by the existence of a one-dimensional subgroup whose orbits are circles. The results proved do not depend on the dimension of the manifold nor on the existence of a metric, but merely on the fact that the Lie group acts globally on the manifold. Firstly some results for the general case of an $m + 1$ -dimensional Lie group are derived: those commutators of the associated Lie algebra involving the generator of the cyclic subgroup, X_0 say, are severely restricted and, in a suitably chosen basis, take a simple form. The Jacobi identities involving X_0 are then applied to show there are further restrictions on the structure of the Lie algebra. All Lie algebras of dimensions 2 and 3 compatible with cyclic symmetry are obtained. In the two-dimensional case the group must be Abelian. For the three-dimensional case, the Bianchi type of the Lie algebra must be I, II, III, VII₀, VIII or IX and furthermore in all cases the vector X_0 forms part of a basis in which the algebra takes its canonical form. Finally four-dimensional Lie algebras compatible with cyclic symmetry are considered and the results are related to the Petrov-Kruchkovich classification of all four-dimensional Lie algebras.

† E-Mail: barnesa@aston.ac.uk Fax: +44 121 333 6215

1. Introduction

Carter (1970) defined a spacetime \mathcal{M} to admit a cyclical symmetry if and only if the metric is invariant under the effective smooth action $SO(2) \times \mathcal{M} \rightarrow \mathcal{M}$ of the one-parameter cyclic group $SO(2)$. A cyclically symmetric spacetime in which the set of fixed points of this isometry is not empty is said to be axially symmetric and the set of fixed points itself is referred to as the axis (of symmetry). Cyclic symmetry is thus a slight generalisation of the concept of axial symmetry. Although the assumption of the existence of an axis is reasonable in many circumstances, there are physically realistic situations where an axis may not exist: the 'axis' may be singular and so not part of the manifold, or the topology of the manifold may be such that no axis exists. An example of the latter situation is a torus embedded in a 3-dimensional flat space; the torus is cyclically symmetric but the Killing vector field generating the rotation does not vanish anywhere on the surface.

Mars and Senovilla (1993) proved a number of useful results on the structure of the axis if it exists. Carot, Senovilla and Vera (1999) showed in an axial symmetric spacetime that if there is a second Killing vector which, with the Killing vector generating the axial symmetry, generates a two-dimensional isometry group then the two Killing vectors commute and the isometry group is Abelian. A similar result for stationary axisymmetric spacetimes was proved by Carter (1970). The proofs of both results depend heavily on the existence of an axis. It is interesting therefore to consider whether a result of the same ilk as those of Carter and Carot et al. are valid for cyclically symmetric spacetimes. Bičák & Schmidt (1984) showed that if in cyclically symmetric spacetime there is a second Killing vector which, with the Killing vector generating the cyclic symmetry, generates a two-dimensional isometry group then the two Killing vectors commute. As pointed out by Barnes (2000) this result does not depend on the dimension of the manifold, nor on the fact that the transformation group is an isometry group nor on the existence of a non-empty axis. Thus the following proposition is valid: any two-dimensional Lie transformation group acting on a manifold which contains a one-dimensional subgroup whose orbits are circles, must be Abelian. The method used to prove this result was extended by Barnes (2000) to apply to three-dimensional Lie transformation groups and it was shown that the existence of a one-dimensional subgroup with closed orbits restricts the Bianchi type of the associated Lie algebra to be I, II, III, VII₀, VIII or IX.

In this paper we will consider Lie transformation groups of arbitrary (finite) dimension containing a one-dimensional subgroup acting cyclically on the manifold (that is with closed circular orbits) and show that the structure of the group is restricted considerably by this assumption. The term "cyclically symmetric manifold" will be used to refer to this situation which is thus being used in a less restricted sense than that of Carter (1970) which referred only to isometry groups. Therefore the results below may be applied not just to isometries, but to more general symmetries: homothetic and conformal motions, affine, projective and curvature collineations etc..

Furthermore the results do not depend the existence of an axis nor on the dimension of the manifold nor indeed on the existence of a metric.

2. Cyclically symmetric manifolds admitting a G_{m+1}

Suppose \mathcal{M} is an n dimensional manifold which admits an $m+1$ dimensional Lie transformation group G_{m+1} acting on \mathcal{M} which contains a one-dimensional subgroup isomorphic to $SO(2)$. Suppose X_0 is the vector field generating this subgroup. We restrict attention to the open submanifold \mathcal{N} of \mathcal{M} on which X_0 is non-zero. The orbit each point of \mathcal{N} under the cyclic symmetry is a circle and these circles are the integral curves of the vector field X_0 . Let ϕ be a circular coordinate running from 0 to 2π which labels the elements of $SO(2)$ in the usual way. Then we can introduce a system of coordinates x^i with $i = 1 \dots n$ and $x^1 = \phi$ adapted to X_0 such that $X_0 = \partial_\phi$. These coordinates are determined only up to transformations of the form

$$\tilde{\phi} = \phi + f(x^\nu) \quad \tilde{x}^\mu = g^\mu(x^\nu)$$

where f and g^μ are smooth functions and where Greek indices take values in the range $2 \dots n$.

Let X_a be vector fields on \mathcal{M} which, together with X_0 , form a basis of the Lie algebra of G_{m+1} . Here and below indices a, b and c take values in the range $1 \dots m$. The commutators involving X_0 may be written in the form

$$[X_0 \ X_a] = A_a^b X_b + B_a X_0 \quad (1)$$

where A_a^b and B_a are constants and the Einstein summation convention has been used over the range $1 \dots m$. If we introduce new basis vectors \tilde{X}_a given by $\tilde{X}_a = P_a^b X_b$, where P_a^b are constants, the structure constants transform as follows

$$\tilde{\mathbf{A}} = \mathbf{P} \mathbf{A} \mathbf{P}^{-1} \quad \tilde{\mathbf{B}} = \mathbf{P} \mathbf{B} \quad (2)$$

where for conciseness standard matrix notation has been used. Using these transformations the matrix \mathbf{A} may be reduced to Jordan normal form. In what follows we will work in a basis in which the structure constants A_a^b are in Jordan normal form but, for typographic simplicity, tildes will be omitted.

Let X be a Jordan basis vector which is a real eigenvector of \mathbf{A} with corresponding real eigenvalue λ , the commutation relation (1) for X is thus

$$[X_0 \ X] = \lambda X + BX_0 \quad (3)$$

where B is a constant. In terms of a coordinate system adapted to X_0 in which $X_0 = \partial_\phi$, the commutation relation becomes

$$\frac{\partial X^i}{\partial \phi} = \lambda X^i + B \delta_0^i$$

where i ranges over $1 \dots n$. If the solutions of these equations are to be periodic in ϕ , then the eigenvalue λ must zero and moreover $B = 0$ in (3).

Suppose now that $Z = X + iY$ (where X and Y are real vectors) is an eigenvector of \mathbf{A} with corresponding eigenvalue $\lambda + i\mu$, the commutation relation (1) for Z reduces to

$$[X_0 \ X] = \lambda X - \mu Y + BX_0 \quad [X_0 \ Y] = \lambda Y + \mu X + CX_0 \quad (4)$$

where B and C are real constants. In terms of an adapted coordinate system in which $X_0 = \partial_\phi$, these equations reduce to

$$\frac{\partial X^i}{\partial \phi} = \lambda X^i - \mu Y^i + B\delta_0^i \quad \frac{\partial Y^i}{\partial \phi} = \lambda Y^i + \mu X^i + C\delta_0^i$$

where i takes values in the range $1 \dots n$. If the solutions of these equations are to be periodic in ϕ with period 2π , then $\lambda = 0$ and $\mu = \pm N$ where N is a positive integer. Furthermore by redefining X and Y according to

$$\tilde{X} = X + C\mu^{-1}X_0 \quad \tilde{Y} = Y - B\mu^{-1}X_0$$

we may set $B = C = 0$ in the commutation relations (4).

Thus we have shown that the only eigenvalues of \mathbf{A} are zero or of the form $\pm iN$ where N is an integer. Furthermore all of the Jordan blocks must be simple or equivalently the minimal polynomial of \mathbf{A} must have no repeated factors. To see this suppose first that there is a non-simple elementary divisor with eigenvalue zero. Thus there is a vector X such that $\mathbf{A}^2X = 0$ but $\mathbf{A}X \equiv Y \neq 0$. Thus we have

$$[X_0 \ X] = Y + BX_0 \quad [X_0 \ Y] = 0$$

where B is a constant. In a coordinate system in which $X_0 = \partial_\phi$, these reduce to

$$\frac{\partial X^i}{\partial \phi} = Y^i + B\delta_0^i \quad \frac{\partial Y^i}{\partial \phi} = 0$$

The solution X of these differential equations is clearly linear in ϕ as $Y \neq 0$. However the components of the vectors X and Y must be periodic in ϕ , which is a contradiction. A similar argument shows that there can be no non-simple elementary divisors of \mathbf{A} corresponding to the purely imaginary eigenvalues.

Thus without loss of generality, we may suppose that \mathbf{A} has p eigenvalues for some p with $0 \leq 2p \leq m$ of the form in_j where each n_j is a positive integer and $1 \leq j \leq p$ with corresponding complex eigenvectors $Z_j \equiv X_{2j} + iX_{2j-1}$ plus $m - 2p$ zero eigenvalues with corresponding real eigenvectors X_k where $2p + 1 \leq k \leq m$. Choosing these m vectors X_a as the basis vectors, the commutators become

$$[X_0 \ X_{2j-1}] = n_j X_{2j} \quad \text{for } 1 \leq j \leq p \quad (5a)$$

$$[X_0 \ X_{2j}] = -n_j X_{2j-1} \quad \text{for } 1 \leq j \leq p \quad (5b)$$

$$[X_0 \ X_k] = 0 \quad \text{for } 2p + 1 \leq k \leq m \quad (5c)$$

If the n_j 's are distinct, the remaining basis freedom preserving the commutation relations (5) is a rotation and dilatation in each of the p two-planes generated by the pairs of vectors X_{2j-1} and X_{2j} for $1 \leq j \leq p$ plus a general (non-singular) transformation of the vectors X_k for $2p+1 \leq k \leq m$:

$$\tilde{X}_{2j-1} = C_j(\cos \theta_j X_{2j-1} + \sin \theta_j X_{2j}) \quad (6a)$$

$$\tilde{X}_{2j} = C_j(-\sin \theta_j X_{2j-1} + \cos \theta_j X_{2j}) \quad (6b)$$

$$\tilde{X}_k = A_k^l X_l + B_k X_0 \quad (6c)$$

where A_k^l and B_k are constants for $k, l = 2p+1 \dots m$, as are C_j and θ_j for $j = 1 \dots p$. If the n_j 's are not all distinct there is some additional freedom in the choice of the associated pairs of basis vectors.

We now consider the restrictions imposed on the commutators by those Jacobi identities which involve the generator X_0 of the cyclic subgroup

$$[X_0 [X_i X_j]] + [X_i [X_j X_0]] + [X_j [X_0 X_i]] = 0 \quad (7)$$

where $i, j = 1 \dots m$. If $p = 0$, that is if all the vectors commute with X_0 , these Jacobi identities yield no information. For $p \neq 0$ there are four cases to consider as follows.

(i) If X_i and X_j both commute with X_0 , we have $[X_0 [X_i X_j]] = 0$ and hence the structure constants satisfy

$$C_{ij}^k = 0 \quad \text{for } 1 \leq k \leq 2p \quad \text{and} \quad 2p+1 \leq i, j \leq m \quad (8)$$

(ii) If X_{2k-1} and X_{2k} are the basis vectors associated with the imaginary eigenvalue in_k , then we have $[X_0 [X_{2k-1} X_{2k}]] = 0$ and thus

$$C_{2k-1 2k}^i = 0 \quad \text{for } 1 \leq i \leq 2p \quad \text{and} \quad 1 \leq k \leq p \quad (9)$$

(iii) If X_j commutes with X_0 but X_i does not, then we have

$$[X_0 [X_{2k-1} X_j]] = n_k [X_{2k} X_j] \quad \text{for } i = 2k-1$$

$$[X_0 [X_{2k} X_j]] = -n_k [X_{2k-1} X_j] \quad \text{for } i = 2k$$

where $1 \leq k \leq p$ and $2p+1 \leq j \leq m$. Thus immediately we can deduce

$$C_{ij}^0 = C_{ij}^l = 0 \quad \text{for } 1 \leq i \leq 2p, \quad 2p+1 \leq j, l \leq m \quad (10)$$

Also we have

$$n_l C_{2k-1 j}^{2l-1} = n_k C_{2k j}^{2l} \quad - n_l C_{2k-1 j}^{2l} = n_k C_{2k j}^{2l-1} \quad (11a)$$

$$n_l C_{2k j}^{2l-1} = -n_k C_{2k-1 j}^{2l} \quad - n_l C_{2k j}^{2l} = -n_k C_{2k-1 j}^{2l-1} \quad (11b)$$

where $1 \leq k, l \leq p$ and $2p+1 \leq j \leq m$. Thus

$$C_{2k-1 j}^{2k-1} = C_{2k j}^{2k} \quad C_{2k-1 j}^{2k} = C_{2k j}^{2k-1} \quad (12)$$

for $1 \leq k \leq 2p$ and $2p+1 \leq j \leq m$. For $k \neq l$, eliminating $C_{2k-1,j}^{2l-1}$ and $C_{2k,j}^{2l-1}$ from (11) we have

$$(n_l^2 - n_k^2)C_{2k-1,j}^{2l} = 0 \quad (n_l^2 - n_k^2)C_{2k,j}^{2l} = 0$$

Since all n_i 's are positive, if $n_l \neq n_k$ we have

$$C_{2k-1,j}^{2l-1} = C_{2k,j}^{2l-1} = C_{2k-1,j}^{2l} = C_{2k,j}^{2l} = 0 \quad \text{for } l \neq k \quad (13)$$

where $1 \leq k, l \leq p$ and $2p+1 \leq j \leq m$.

(iv) If the basis vectors are associated with different imaginary eigenvalues n_i and n_j then

$$[X_0 \ [X_{2i-1} \ X_{2j-1}]] = n_j[X_{2i-1} \ X_{2j}] + n_i[X_{2i} \ X_{2j-1}] \quad (14a)$$

$$[X_0 \ [X_{2i-1} \ X_{2j}]] = -n_j[X_{2i-1} \ X_{2j-1}] + n_i[X_{2i} \ X_{2j}] \quad (14b)$$

$$[X_0 \ [X_{2i} \ X_{2j-1}]] = n_j[X_{2i} \ X_{2j}] - n_i[X_{2i-1} \ X_{2j-1}] \quad (14c)$$

$$[X_0 \ [X_{2i} \ X_{2j}]] = -n_j[X_{2i} \ X_{2j-1}] - n_i[X_{2i-1} \ X_{2j}] \quad (14d)$$

It follows immediately that

$$n_j C_{2i-1,2j}^k + n_i C_{2i,2j-1}^k = 0 \quad -n_j C_{2i-1,2j-1}^k + n_i C_{2i,2j}^k = 0 \quad (15a)$$

$$n_j C_{2i,2j}^k - n_i C_{2i-1,2j-1}^k = 0 \quad -n_j C_{2i,2j-1}^k - n_i C_{2i-1,2j}^k = 0 \quad (15b)$$

If the integers n_i are distinct we may deduce

$$C_{2i-1,2j-1}^k = C_{2i-1,2j}^k = C_{2i,2j-1}^k = C_{2i,2j}^k = 0 \quad (16)$$

for $1 \leq i, j \leq p$ and $k = 0$ or $2p+1 \leq k \leq m$ with $i \neq j$. Also we obtain two sets each of four linear equations for four structure constants:

$$\begin{pmatrix} n_h & 0 & -n_i & -n_j \\ 0 & n_h & n_j & n_i \\ -n_i & n_j & n_h & 0 \\ -n_j & n_i & 0 & n_h \end{pmatrix} \begin{pmatrix} C_{2i-1,2j-1}^{2h-1} \\ C_{2i,2j}^{2h-1} \\ C_{2i-1,2j}^{2h} \\ C_{2i,2j-1}^{2h} \end{pmatrix} = \mathbf{0} \quad (17a)$$

$$\begin{pmatrix} n_h & 0 & n_i & -n_j \\ 0 & n_h & n_j & -n_i \\ n_i & n_j & n_h & 0 \\ -n_j & -n_i & 0 & n_h \end{pmatrix} \begin{pmatrix} C_{2i-1,2j}^{2h-1} \\ C_{2i,2j-1}^{2h-1} \\ C_{2i-1,2j-1}^{2h} \\ C_{2i,2j}^{2h} \end{pmatrix} = \mathbf{0} \quad (17b)$$

where $1 \leq h, i, j \leq p$ with $i \neq j$. If $n_h \neq \pm n_i \pm n_j$, that is if no integer n_k is a sum of two other such integers, the matrices in (17a,b) are non-singular and it follows that all of these eight structure constants are zero. Thus with the aid of equation (9) we may deduce

$$C_{ij}^k = 0 \quad \text{for } 1 \leq i, j, k \leq 2p \quad (18)$$

Thus for $p > 0$, we have shown that the Jacobi identities imply that a number of blocks of the structure constants vanish as given by equations (8), (9) and (10) and in addition equation (12) is always satisfied. Also for the generic case where the p integers n_i are distinct and no integer n_i is the sum of two other such integers even more of the structure constants vanish as given by Eqs. (13), (16) and (18).

Even if the case where $n_i = n_j$ for some $i \neq j$, there are still restrictions on the structure constants imposed by the Jacobi identities; the solutions of equations (11) and (15) are respectively

$$C_{2k-1}^{2l-1} = C_{2k}^{2l} \quad C_{2k}^{2l-1} = -C_{2k-1}^{2l} \quad (19)$$

for $1 \leq k, l \leq p$ and $2p+1 \leq j \leq m$ and

$$C_{2i-1}^k C_{2j-1}^k = C_{2i}^k C_{2j}^k \quad C_{2i-1}^k C_{2j}^k = -C_{2i}^k C_{2j-1}^k \quad (20)$$

for $1 \leq i, j \leq p$ and $k = 0$ or $2p+1 \leq k \leq m$. If $n_h = n_i + n_j$, the solutions of equations (17a,b) are

$$C_{2i-1}^{2h-1} = -C_{2i}^{2h-1} = C_{2i-1}^{2h} = C_{2i}^{2h-1} \quad (21a)$$

$$C_{2i-1}^{2h-1} = C_{2i}^{2h-1} = -C_{2i-1}^{2h} = C_{2i}^{2h} \quad (21b)$$

whereas if $n_h = n_i - n_j$ the solutions are

$$-C_{2i-1}^{2h-1} = -C_{2i}^{2h-1} = -C_{2i-1}^{2h} = C_{2i}^{2h-1} \quad (22a)$$

$$-C_{2i-1}^{2h-1} = C_{2i}^{2h-1} = C_{2i-1}^{2h} = C_{2i}^{2h} \quad (22b)$$

and for $n_h = n_i - n_j$ the solutions are obtained from (22) by switching i and j .

The structure constants must also satisfy the Jacobi identities

$$[X_i [X_j X_k]] + [X_j [X_k X_i]] + [X_k [X_i X_j]] = 0 \quad (23)$$

where $i, j, k = 1 \dots m$. However in the case general these do not lead to particularly simple relations and so will not be considered at this stage. However this identity is used in the analysis of the four dimensional Lie groups below.

3. Cyclically symmetric manifolds admitting a G_2 or G_3

Before considering the four dimensional case the results of the previous section are applied to the two and three-dimensional cases. For the two-dimensional case ($m = 1$) equation (5) reduces to $[X_0 X_1] = 0$ and so the group is Abelian which is the result of Bičák & Schmidt (1984).

For the three-dimensional case ($m = 2$) equation (5) allows two possibilities: either X_1 and X_2 commute with X_0 or we have $[X_0 X_1] = nX_2$ and $[X_0 X_2] = -nX_1$ for some positive integer n . In the former case the Jacobi identities are identically satisfied and

by a basis transformation of the form (6c) we may reduce the commutation relations to one of three following algebraically distinct forms

$$\begin{array}{llll}
 \text{Bianchi type I} & [X_0 X_1] = 0 & [X_0 X_2] = 0 & [X_1 X_2] = 0 \\
 \text{Bianchi type II} & [X_0 X_1] = 0 & [X_0 X_2] = 0 & [X_1 X_2] = X_0 \\
 \text{Bianchi type III} & [X_0 X_1] = 0 & [X_0 X_2] = 0 & [X_1 X_2] = X_2
 \end{array}$$

In the latter case equation (9) implies $C_{12}^1 = C_{12}^2 = 0$ and so $[X_1 X_2] = aX_0$ where a is a constant. By using the freedom to rescale X_1 and X_2 as in equations (6a,b) we may set $a = 0, \pm 1$ and so arrive at one of the three algebraically distinct forms

$$\begin{array}{llll}
 \text{Bianchi type VII}_0 & [X_0 X_1] = nX_2 & [X_0 X_2] = -nX_1 & [X_1 X_2] = 0 \\
 \text{Bianchi type VIII} & [X_0 X_1] = nX_2 & [X_0 X_2] = -nX_1 & [X_1 X_2] = -X_0 \\
 \text{Bianchi type IX} & [X_0 X_1] = nX_2 & [X_0 X_2] = -nX_1 & [X_1 X_2] = X_0
 \end{array}$$

We may set $n = 1$ in the second set of commutators by rescaling X_0 and so obtain the canonical forms of the commutation relations for Bianchi types VII_0 , VIII and IX . However, if this is done, we cannot maintain both the 2π periodicity of the coordinate ϕ and the relation $X_0 = \partial_\phi$.

4. Cyclically symmetric manifolds admitting a G_4

For the four-dimensional case ($m = 3$) equation (5) allows two possibilities: either X_1 , X_2 and X_3 all commute with X_0 or we have

$$[X_0 X_1] = nX_2 \quad [X_0 X_2] = -nX_1 \quad [X_0 X_3] = 0 \quad (24)$$

for some positive integer n .

In the former case the Jacobi identities involving X_0 are identically satisfied. Thus the cyclic symmetry does not imply further restrictions on the structure constants in this case. The basis vectors X_1 , X_2 and X_3 are only defined up to the transformations of the form

$$\tilde{X}_k = A_k^l X_l + B_k X_0 \quad (25)$$

where A_k^l and B_k are constants for $k, l = 1 \dots 3$. The Jacobi identity (23) remains to be satisfied; this leads to the following restrictions on the structure constants

$$C_{1l}^\alpha C_{23}^l + C_{2l}^\alpha C_{31}^l + C_{3l}^\alpha C_{12}^l = 0 \quad (26)$$

where $l = 1 \dots 3$ and $\alpha = 0 \dots 3$. The following classification method may then be used: choose A_k^l in equation (25) so that C_{jk}^i take one of the nine canonical (Bianchi) forms for a three-dimensional Lie algebra; then use equation (26) with $\alpha = 0$ and the freedom in the choice of the basis vectors in equation (25) (choice of B_k and any

remaining freedom in the choice of A_k^l) to simplify C_{ij}^0 as far as possible. Although the analysis is straightforward, it will not be presented here as more than a dozen separate cases need to be considered.

In the latter case where equation (24) is satisfied, the Jacobi identities involving X_0 imply that

$$[X_1 X_2] = aX_3 + bX_0 \quad [X_2 X_3] = cX_1 + dX_2 \quad [X_3 X_1] = -dX_1 + cX_2$$

where a, b, c and d are constants. However, without loss of generality, we may set $c = 0$ by means of the basis change $\tilde{X}_3 = X_3 - c/nX_0$. Furthermore the remaining Jacobi identity (23) implies $d(aX_3 + bX_0) = 0$ so that either $d = 0$ or $a = b = 0$.

If $d \neq 0$, we may set $d = 1$ by rescaling X_3 and the commutation relations take the form

$$\begin{aligned} [X_0 X_1] &= nX_2 & [X_0 X_2] &= -nX_1 & [X_0 X_3] &= 0 \\ [X_1 X_2] &= 0 & [X_2 X_3] &= X_2 & [X_3 X_1] &= -X_1 \end{aligned}$$

which, apart from the numbering and signs of the basis vectors, is the canonical form for Kruchkovich-Petrov class V (Kruchkovich, 1954 & 1957, Petrov 1963). There is no three-dimensional sub-algebra containing X_0 , but clearly there is a sub-algebra spanned by X_1, X_2 and X_3 .

If $d = 0$, then the commutation relations take the form

$$\begin{aligned} [X_0 X_1] &= nX_2 & [X_0 X_2] &= -nX_1 & [X_0 X_3] &= 0 \\ [X_1 X_2] &= aX_3 + bX_0 & [X_2 X_3] &= 0 & [X_3 X_1] &= 0 \end{aligned}$$

Note that X_3 generates the centre of the Lie algebra, that is it is the only vector that commutes with all the vectors of the algebra. There are essentially only six algebraically distinct cases:

- (i) If $a = b = 0$, X_1, X_2 and X_3 generate an abelian sub-algebra and so the Lie algebra is of Kruchkovich-Petrov type VI; in fact it is of type VI₄ with $k = l = 0$ (Petrov 1963). This algebra is a central extension by X_3 of the Bianchi type VII₀ algebra in section 3. Here and in classes (ii)-(iv) below, the vectors X_0, X_1, X_2 and X_3 form a canonical basis of the Lie algebra.
- (ii) If $a = 0$ and $b = +1$, the algebra is of Kruchkovich-Petrov type VIII. This algebra is a central extension by X_3 of the Bianchi type IX algebra in section 3.
- (iii) If $a = 0$ and $b = +1$, the algebra is of Kruchkovich-Petrov type VII. This algebra is a central extension by X_3 of the Bianchi type VIII algebra in section 3.
- (iv) If $a = 1$ and $b = 0$, the algebra is of Kruchkovich-Petrov type III_{q=0}. There is no three-dimensional sub-algebra containing X_0 , but clearly there is a sub-algebra spanned by X_1, X_2 and X_3 .

(v) If $a = b = 1$, the algebra is again of Kruchkovich-Petrov type VIII, but unlike class (ii), X_0 is not a canonical basis vector. Instead in this case a canonical basis of the algebra is $X_0 + X_3$, X_1 , X_2 and X_3 .

(vi) If $a = 1$ and $b = -1$, the algebra is of Kruchkovich-Petrov type VII, but unlike class (iii), X_0 is not a canonical basis vector. Instead in this case a canonical basis of the algebra is $X_0 - X_3$, X_1 , X_2 and X_3 .

In classes (v) and (vi), there is no three-dimensional sub-algebra containing X_0 , but clearly there is a sub-algebra spanned by X_1 , X_2 and $X_0 \pm X_3$ which is of Bianchi type IX or VIII in cases (v) and (vi) respectively.

5. Summary

Lie transformation groups which act on an n -dimensional manifold \mathcal{M} and which contain a one-dimensional subgroup acting cyclically on \mathcal{M} have been considered. It is found that the existence of a cyclic action imposes quite severe restrictions on the structure constants of the group.

Such two-dimensional groups must be Abelian and for the three-dimensional case the Bianchi type of the group is restricted: types IV, V and VI cannot occur and only the subclass $\text{VII}_{q=0}$ of type VII is allowed. The results for the three-dimensional case were obtained by the author (Barnes, 2000).

At first sight the restrictions in the three-dimensional case may not seem too severe: only three of the nine Bianchi types are excluded whilst a fourth type is restricted. However, it should be pointed out that Bianchi types VI and VII both involve an arbitrary parameter and so each contain an infinite number of algebraically distinct types. Those in type VI are excluded completely and for type VII only a single case survives. Moreover for most of the types which *are* permitted the structure of the Lie algebra is ‘aligned nicely’ relative to X_0 . For example, for Bianchi type II, X_0 is also the generator of the first derived subalgebra and for Bianchi types II and III, X_0 lies in the centre of the algebra. For Bianchi types VII_0 , VIII and IX, X_0 is a member of a canonical basis of the Lie algebra.

The structure of four-dimensional groups is again severely restricted by the existence of a one-dimensional subgroup acting cyclically on \mathcal{M} . In the case when the generator of the cyclic action is not in the centre of the algebra, only five of the classes of Kruchkovich (1954, 1957) and Petrov (1969) can occur: namely III, IV_4 , V, VII and VIII, with only zero-parameter sub-cases of types III and IV_4 occurring. Moreover in all cases X_0 is ‘aligned nicely’ relative to a canonical basis of the Lie algebra.

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